Gradient representations of the Newtonian fluid

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Abstract

We propose a model of gradient representations for study of the Newtonian fluid. In particular, we consider the possibility of smooth Navier-Stokes solutions defined by a particular solution ψ. This make it possible for us to research the Newtonian fluid representation described by the Navier-Stokes equations. We also consider the smoothness characteristics of an Navier-Stokes solution which admit such representation. In conclusion, we extend our theory and consider its complements to the L^p-theory.

1 Introduction

The subject of this study is the existence of Navier–Stokes solutions 1 [2]. However, despite the achievements in researches of weak boundaries [1] and the difficulties in finding smooth solutions [3], it seems that such possibility of the existence is already interesting for us. It is important to note that in this paper we only research the possibility of existence and not the characteristics of an smoothness by itself.

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \nabla u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \quad (x \in \mathbb{R}^n, t \geq 0) \tag{1}$$

For the purposes of our study, we introduce a model of gradient representations relatively to which we consider the possibility of an smooth solutions. This seems doubly interesting, since the very presence of gradient representations opens new perspectives for research smooth solutions independently. Let us define an axiom which we will use as basis for further theoretical structures.

Axiom 1. If ψ is a solution for 1 then $u(x,0) = u^{\circ}(x)$ satisfies the following system of equations:

$$\begin{cases} \operatorname{grad} x = \sum_{i=1}^{n} \dfrac{\partial \psi}{\partial x_i} & (x \in \mathbb{R}^n, t \geq 0) \\ \operatorname{div} u = \sum_{i=1}^{n} \dfrac{\partial u_i}{\partial x_i} = 0 & (x \in \mathbb{R}^n, t \geq 0) \end{cases} \tag{2}$$

Undoubtedly, regardless of the gradient representations model's importance for the mathematical researches, we can not bypass the question of physical significance for our model. However, in trying to answer this question, it would be necessary for us consider the very meaning of the physical. We assume that axiom 1 is suitable for us as long as suitable some solution of an differential equation.

At the same time, we have no doubt that the model of gradient representations can be verified and significantly expanded using the same theory of differential equations that underlies axiom 1. In this way, we assume possibility to perceive our research by itself and the theory within which it was reached. Respectfully, our model appears to be physical as much as it appears in particular. Let us continue our research by discovering conclusions of solution ψ.

2 Gradient of an particle

Theorem 2.1. *There is no similar particles of an fluid element subject.*

$$\forall a, b \in U \, \exists \, C : a \neq C \wedge b \neq C \quad where \, U = \{u\} \tag{3}$$

Let there is such u_u so some $a, b \in U$ equal each other since a or b equal u_u. In other words, we propose possibility of equality between two elements of U by an quality of solution ψ. Since unique vector u_u presents us such quality, let us declare ψ directly.

$$\psi = \int_U \frac{\operatorname{grad} x}{\sum\limits_{i=1}^{n} \partial x_i} \, du \quad where \, du \sim \operatorname{div} u \tag{4}$$

Obviously there is no direct declaration of ψ, but in case of axiom 1 such declaration can not be not to be. This is at least supposed by presence of $\operatorname{div} u$ without which we lose all axiomatic issues of axiom 1. In this way, direct declaration of ψ anyway has to be as hypothetical condition of solvability for an equation. It could be curious to research declaration 4 numerically, however it is not principal for the purposes of our study right now. Moreover, the Question about essence of certain differential equation's solution seems more like philosophical question other then mathematical, where such essence seems to be a question by itself.

For further research let us prove following lemma.

Lemma 2.2. *All parts of solution ψ located in U.*

$$\frac{\partial u}{\partial \psi} \in U \tag{5}$$

Proof. The proof is trivial. If 2.2 is not true, there is exist some $u \notin U$ which contradict to 1 and definition of ψ. □

2.2 gives us sophisticated possibility to research relation between ψ and u as a constant value.

$$\frac{\partial u}{\partial \psi} = const \tag{6}$$

Intuitively, it seems obviously when we perceive a surface of an dynamical fluid, ocean e.g., as uniform area. In case of lack of ocean and in rest cases, constant characteristic of 6 seems natural as followed from axiom 1 and 2.2. Farther, we denote *const* — result of 2.2 — as letter C and function of relation — in our case division — as operator F.

Now we get a contradiction which proves theorem 2.1 Q.E.D.

$$\begin{gathered} F(a) = u_u \wedge F(b) = u_u \wedge a = b = u_u \; \exists \, F(u_u) : \\ F(a) = a \wedge F(b) = b \wedge F(u_u) = u_u \vee F(a) = C \wedge F(b) = C \wedge F(u_u) \neq C \end{gathered} \tag{7}$$

In addition to the proof of theorem 2.1 at the final of this section, we get significant operator F. Regardless of the division's value which F represented, we have opportunity to research "inner" relation between particles relatively to axiom 1. For that purpose let us proves theorem which expresses set of u as a group.

3 The set of gradient particles

Theorem 3.1. *There is exist a homomorphism for the set, so this set is a group.*

$$\begin{gathered} d(a,b), d(x,y) \in U \; \exists \, F : \\ F(d(a,b)) \sim F(d(x,y)) \wedge d(a,x) \sim d(b,y) \; \forall \, a,b,x,y \in U \end{gathered} \tag{8}$$

Let us prove this theorem by considering u separately. Let us choose such $a, b, x, y \in U$ which both satisfy system 2 and a distance function d on U. Definition d is trivial.

Definition 3.1. $d : U \times U \to [0, \infty)$ where $[0, \infty)$ is the set of non-negative real numbers and for all $x, y, z \in U$, the following conditions are satisfied:

D1 $d(x,y) = 0 \Leftrightarrow x = y$
D2 $d(x,y) = d(y,x)$
D3 $d(x,z) \leq d(x,y) + d(y,z)$

Intuitively, it looks evidently that D3 does not cause problems. However, it looks sophisticated for us to prove it. In particular, that $d(x,z) \leq d(x,y) + d(y,z)$ is fair for every $x, y, z \in U$ or, in other words, that $U = \mathbb{R}$. For further research let us prove following lemma in relation to D3.

Lemma 3.2. *Result of any additions between particles of an fluid element subject related to that fluid.*

$$+ : U \times U \to \{u\} \tag{9}$$

Proof. Let there is such $x \in \{u\}$ so $x + y \notin U \; \forall y \in U$. Since $F(x) = C$ as 6, there is exist such y so $F(x+y) \neq C$. But, since $F(n) = C$ from 2.2 so all $n \in U$ as 6, it is fair to say that all $y \in U$ since $F(y) = C$. So, or for all x is true that $x + y \in U$ or $x \notin \{u\}$ and $x \notin U$, respectfully. Since all u in U and all x, y in U and $x + y \in U$ for all x and y, we can conclude that $+ : U \times U \to \{u\}$ for all x, y and $x + y$ in U. □

If we go back to issue D3, from 3.2 we can assume that $d(x, z) \simeq d(x, y) + d(y, z)$ and $d(x, z) \lesssim d(x, y) + d(y, z)$. Along with it, we get another tricky question: does $a \leq b$ equivalent to $a \simeq b$ and $a \lesssim b$ in our case or not? However, since from solution's definition 4 follows similarity $du \sim \operatorname{div} u$, equality $d(x, z) > d(x, y) + d(y, z)$ leads to contradiction. Indeed, if equality $d(x, z) > d(x, y) + d(y, z)$ would be true along with $du \sim \operatorname{div} u$, there would have to be exists a particle that is similar to itself which contradict to theorem 2.1. Moreover, issue with D3 will not make any sense at all in this case. As it was mentioned before, similarity $du \sim \operatorname{div} u$ in relation to axiom 1 follows by definition ψ, so we can exclude issue with D3 from focus of our research.

Relation $d(a, x) \sim d(b, y)$ obviously follows from 3.2. Indeed, $d(a, b)$, $d(x, y)$ has to be similar because of common affiliation to U. Intuitively, it seems indisputable that it is impossible to "calculate" a fluid without proper scoop. In case of lemma 2.2, similarity between $d(a, x) \sim d(b, y)$ is provided by common ψ as 2.2. In the same way, similarity is ensured between $d(b, x)$ and $d(a, y)$.

Finally, we can complete our prove. Since there is no similar particles as followed from 2.1 and operator F takes only constant values, distances of particles has to be similar to constant. Indeed, if similarity of distances is not remain constant then operator F could not be applied for each particle since distances of all particles has to be similar by definition 3.1. In accord to 6 , operator F express similarity of distances between particles as a constant value. Q.E.D.

Nature of *const* or C represents significant problem in relation to axiom 1. Throw proving theorems 2.1 and 3.1 we declare C as a self-dependent structure so actual value of it is still open for questions. Especially, is C valuable by itself or only in context of presence solution ψ? Indeed, if u satisfies to 2 only in presence of ψ and as we convinced about C in 6, constant value of C could be constant only hypothetically and actually mean anything since there is no valuable declaration 4 of ψ. To solve such ontological problem, let us continue our discovering to research such valuable quality of C. Convincing prove for us could be possibility to elucidate relation between U and set of natural numbers $\mathbb{N}$. Sets, constancy of which occurs for us constant by definition.

4 Fluid formation of gradient particles

Theorem 4.1. *The group U is an infinite periodic group.*

$$\forall a \in U \; \exists \, n \in \mathbb{N} : \operatorname{ord} a = n \tag{10}$$

Let us choose two elements a, b of U for which satisfies equality $F(a) = C$ and $F(b) = C$ and prove following lemma.

Lemma 4.2. *a, b are contained in U only and only if their ordinal numbers not equal.*

$$a, b \in U \; \exists \, n, m \in \mathbb{N} \wedge \operatorname{ord} a = n \wedge \operatorname{ord} b = m \wedge n \neq m \tag{11}$$

Proof. Let us choose such u_0 for which $F(u_0) = C$ and consider $\operatorname{ord} u_0 = 0$. Obviously from 6, if $F(u_n) = C$ for all $n \in \mathbb{N}$ then $u_n \in U$. Since that, $\operatorname{ord} u_n \in \mathbb{N}$ as long as $u_n \in U$ for all $n \in \mathbb{N}$ so $F(u_n) = C$. Finally, if $\operatorname{ord} u_n = \operatorname{ord} u_m$ then $u_n = u_m$ for some $n, m \in \mathbb{N}$ since $F(u_n) = F(u_m)$ and $u_n, u_m \in U$, which contradict to theorem 2.1. □

The above lemma could be generalized as idea that space with constant characteristic should have common point between each two elements of that space which measure up to that constant characteristic. It propose sophisticated possibilities to research fluid element subject as countable object in a topological therms. Intuitively, it seems obviously to understand fluid object as subject all elements of which has topological structure. Since we agree that solution of some different equation solved such equation for each parts as in axiom 1, we could propose such observation formally.

Since we know that all elements of U has no equal ordinal numbers, finishing of prove is trivial. If a, b satisfies equality $F(a) = C$ and $F(b) = C$ and since their ordinal numbers are not equal then for each elements of U should be exists ordinal number in $\mathbb{N}$. Otherwise, there should be exist such ordinal number which element of U not satisfy to relation 6. Q.E.D.

$$\operatorname{ord} a = n \wedge \operatorname{ord} b = n \,\exists\, F : F(a) = C \vee F(b) = a \tag{12}$$

At firs glance, result of proving 4.1 repeats theorem 2.1. Indeed, just like we prove that there is no similar particles, we prove that there is no similar ordinals. But now we have very important relation between U and $\mathbb{N}$. Specifically, that $F : U \to \mathbb{N}$ satisfies theorem 4.1. It gives us possibilities to research variations of $\mathbb{N}$ relatively to U. In other words, we can research relation between various solutions of 1 which could be represented as natural numbers and solution of 1 which has such representation.

We will leave physical meaning of such representation beyond the scope of our research. Indeed, it could be naive to propose, in two words, that fluid could be countable only in natural numbers. However, such propose is out of our research also. Remaining in context of the theory denoted by the axiom 1, we only follow the logical premise that a given solution of a differential equation is its solution by definition. Respectfully, we have opportunity to validate the result of our research as long as it corresponds to this logical premise.

5 Presentation of fluid

Theorem 5.1. *Each solution with initial condition contradicts to 4.1.*

$$\begin{gathered} f_a(u) \sim f_b(u) \wedge f(u) \neq F(u) \,\forall\, a, b \in \mathbb{N} \,\exists\, S : \\ f(u) \to S \wedge U \cap S = \varnothing \end{gathered} \tag{13}$$

Let us prove the theorem by reductio ad absurdum so we will show that equality $U \cap S \neq \varnothing$ leads to a contradiction.

Lemma 5.2. *Functions on U are similar only if they are not equal F.*

$$f_a(u) \sim f_b(u) \wedge f_a(u) \neq F(u) \wedge f_b(u) \neq F(u) \,\forall\, a, b \in \mathbb{N} \wedge u \in U \tag{14}$$

Proof. Let us define on U two functions which both correspond theorem 2.1 so each element of the set of values S not equal C. $f_1 : U \to U$ and $f_2 : U \to U$ where $a \neq C \,\forall\, a \in f_1(u)$ and $b \neq C \,\forall\, b \in f_2(u)$. Suppose that $f_1 = F$. If its true then $U = S_1$ since $F(u) = C$ for all u and $f_1(u) = C$ for all u as well. But, in this case $f_2 \in f_1$ as $f_2 : S_1 \to S_2$ and $S_2 \in U$ by definition. If we suppose the same but $f_2 = F$ we will have the same, similar relation $f_1 \in f_2$. So, or both f_1, f_2 on U equals F so this functions are equal by definition or one function f_1 or f_2 on U equal F so due to 3.1 each one similar to another. □

As it was mentioned before S in 13 represents the set of values of an function on U. Intuitively, result of 5.2 could be imagined as a nature fact that small amount of water is wet as mach as any amount of water only because of possibility to be wet. Formally, presence of gradient in 1 and 4 declare such idea. Indeed, only because of possibility to measure a value of each point x relatively to global context u we were able to declare ψ directly. Respectfully, we are able to consider S as an directly declared object since its relation to u defined by operator F.

Formally, non-empty intersection between U and S means that there are exist some u which equal to itself in case of 5.2. Indeed, from theorem 4.1 we knows that for each element of S should be unique ordinal number as set of S consists from elements of U. But, from 5.2 we knows the function f not equal F and since f have all values on U so their should be such element which, in case of non-empty intersection between U and S, has two equal and different ordinal numbers. Statement, which is absurd. Q.E.D.

$$\begin{gathered} u \in S \,\exists\, z \in \mathbb{N} : u = \operatorname{ord} z \to \\ u \in U \cap S \,\exists\, n \in \mathbb{N} : u = \operatorname{ord} n \wedge u = \operatorname{ord} z \vee z = n \end{gathered} \tag{15}$$

Intuitively, such result could be presented as observation that part of fluid which has unique fluid characters become so unique so losing any characteristics of the part. Formally, theorem 5.1 approve absence of any possible smooth solution of Navier–Stokes equations. Indeed, definition of smoothness suppose presence of derivatives of all orders everywhere in the set of definitions, exactly the same presence as similarity between f_n led us to the proof of 5.1. We leave the subject of smoothness nature for further research and consider theory of gradient representation in the light of contemporary research for conclusion.

6 Conclusion

It is easy to notice that group U maybe examined as a domain with Lipschitz boundary. Indeed, since we proved that $F : U \to \mathbb{N}$, operator F can not fail to meet the property of Lipschitz continuous function. In this case, operator F fails to correspond 6 so we get contradiction by definition. Respectfully, theory of gradient representation relatively to Navier–Stokes equations could be extend to L-theory as it was mentioned is the study of Tolksdorf [4].

It is interesting to note, that result of theorem 5.1 especially successfully combined with solvability in the critical space $\mathrm{L}^\infty(0, T; \mathrm{L}^3_\sigma(\Omega))$. Formally, the sequence of continuous function $u : [0, T) \to \mathrm{W}^1_\sigma(\Omega)]$ converges to a mild solution u as similarity between functions f produce semigroup on $L^p_\sigma(\Omega)$. A similar relation we observed in proof 5.2 when F meets the property of Lipschitz continuous function so f turns out to be similar. Such significant accomplishment opens up promising horizons for further research.

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